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A QUANTITATIVE FINITE-DIMENSIONAL KRIVINE THEOREM†

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ABSTRACT

Measure concentration arguments are applied to get a power-type estimate for the dimension of almost l_p subspaces of isomorphs of l_p^n and for the length of almost-symmetric sequences under a nonlinear-type condition.

1. Introduction

A well-known theorem of I. L. Krivine [7] states that l_p is finitely represented in any isomorph of l_p ($1 \leqq p \leqq \infty$). A qualitative finite-dimensional interpretation is that, given $p \geqq 1$, C, k and $\varepsilon > 0$, there is $n = n(C, k, \varepsilon, p)$ so that every n-dimensional space which is C-isomorphic to l_p^n, contains a k-dimensional subspace $(1+\varepsilon)$-isomorphic to l_p^k. A quantitative version, i.e., an estimate for $n(C, k, \varepsilon, p)$, is given in [12] in the case $1 < p < 2$, namely: $n(C, k, \varepsilon, p) \leqq \exp(\varphi_{\varepsilon,p} C^p k^{p-1})$. Pisier remarks there that a better estimate follows from the results of [2]. He also remarks that the better results known for the cases $p = 1, 2$ and ∞ suggest that the estimate in [12] is not the "right" one.

In this note we want to point out how a power-type estimate which in some sense is the best possible, holding for *all* $1 \leqq p < \infty$, can be deduced from the results of [2], i.e. by using measure concentration phenomena.

In Section 2 we present the general facts from [2] about measure concentration in a somewhat more systematic way and with some modifications due to more recent results in this area (mainly [13]).

Section 3 deals with the finite-dimensional Krivine theorem. The application of measure concentration to the existence of $(1+\varepsilon)$-symmetric sequences under

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a "nonlinear type condition" is discussed in Section 4. (A sequence $(x_1, \ldots, x_n)$ in a normed space is "K-symmetric" if $\|\sum_{i=1}^n \varepsilon_i \alpha_i x_{\pi(i)}\| \leqq K \|\sum_{i=1}^n \alpha_i x_i\|$ for every choice of scalars $\alpha_1, \ldots, \alpha_n$, every choice of signs $\varepsilon_i = \pm 1$ and every permutation π of $\{1, \ldots, n\}$.)

Section 5 is a corrigendum of Theorem 2.5 in [2] which contained a computation mistake.

2. Normal Levy families (measure concentration phenomena)

2.1. By a *normalized metric probability space* (n.m.p.s) we mean a metric space (Ω, d) with diameter 1, and a Borel probability measure μ on (Ω, d). For the n.m.p.s. (Ω, d, μ) we define the *Levy function*

$$\alpha_\Omega(\delta) = \sup\{1 - \mu(A_\delta);\ A \subset \Omega,\ \mu(A) \geqq \tfrac{1}{2}\}, \quad \text{where } A_\delta = \{t \in \Omega;\ d(t, A) \leqq \delta\}.$$

We call $(\Omega_n)_{n=1}^\infty$ a τ*-normal Levy family* if $\alpha_{\Omega_n}(\delta) \leqq \alpha_0 e^{-\tau n \delta^2}$ for some α_0 and for every $\delta > 0$, $n = 1, 2, \ldots$.

2.2. We shall list now several known normal Levy families:

(i) Levy's classical isoperimetric inequality, after normalization, yields that if S_{n-1} is the unit sphere in the n-dimensional Euclidean space with the normalized geodesic distance and the normalized Lebesgue measure, then $(S_{n-1})_{n=1}^\infty$ is a $\frac{1}{2}\pi^2$-normal Levy family (with $\alpha_0 = 1$) (cf. [10] or [3]).

(ii) The Gromov isoperimetric inequality implies (cf. [4]) that if Ω_n is the product space $(S_{k-1})^m$ with the normalized l_2-sum metric

$$d_2^m(s, t) = m^{-1/2} \left(\sum_{i=1}^m d_i(s_i, t_i)^2 \right)^{1/2}$$

and with the product measure, where $mk \geqq n$, then (Ω_n) is a $\frac{1}{2}\pi^2$-normal Levy family (again, with $\alpha_0 = 1$).

In the following discrete examples we assume that the finite space Ω carries the equidistributed probability $\mu(A) = |A|/|\Omega|$.

(iii) If $E_2^n = \{-1, 1\}^n$ has the normalized Hamming metric

$$d(s, t) = \frac{1}{n} |\{i;\ s_i \neq t_i\}|,$$

the isoperimetric inequality for this space (cf. (1.3) in [2]) yields that $(E_2^n)_{n=1}^\infty$ is a 2-normal Levy family (with $\alpha_0 = \frac{1}{2}$).

(iv) If Π_n is the group of permutations of $\{1, \ldots, n\}$ with the normalized Hamming metric

$$d(\pi,\pi')=\frac{1}{n}\,|\{i;\pi(i)\neq\pi'(i)\}|$$

then, as shown by Maurey [9], (Π_n) is a $\frac{1}{64}$-normal Levy family (with $\alpha_0=1$).

Both E_2^n and Π_n are particular cases of a class of n.m.p.s. introduced by Schechtman [13] who analyzed Maurey's proof: Denote by $\mathcal{S}(c,n)$ the class of finite metric spaces (Ω,d) of diameter 1 such that there is a chain $D^0,D^1,\dots,D^n$ of decompositions of Ω, with each D^k $(k=1,2,\dots,n)$ refining D^{k-1}, $D^0=\{\Omega\}$ and $D^n=\{\{t\};t\in\Omega\}$ are the trivial extreme decompositions, and such that whenever $A_i^k,A_j^k\in D^k$ are both subsets of the same $A_m^{k-1}\in D^{k-1}$, there is a one-to-one mapping $\psi_{i,j}^k$ of A_i^k onto A_j^k with $d(\psi_{i,j}^k t,t)\leqq c/n$ for every $t\in A_i^k$.

Schechtman's results mean that if $\Omega_n\in\mathcal{S}(c,n)$, $n=1,2,\dots$, then $(\Omega_n)_{n=1}^\infty$ is $1/16c^2$-normal Levy family (with $\alpha_0=1$). Other examples of this class are:

(v) $\Omega=E_2^{hm}=\Pi_h^m$ with the normalized Hamming metric

$$d((\boldsymbol{\varepsilon},\boldsymbol{\pi}),(\boldsymbol{\varepsilon}',\boldsymbol{\pi}'))=\frac{1}{hm}\,|\{(i,j);1\leqq i\leqq m,1\leqq j\leqq h,(\varepsilon_i(j),\pi_i(j))\neq(\varepsilon_i'(j),\pi_i'(j))\}|$$

is in $\mathcal{S}(2,hm)$. Indeed, taking D^k to be the decompositions into the sets determined by the first $k=(i-1)h+j$ $(1\leqq i\leqq m,\ 1\leqq j\leqq h)$ coordinates, if $A_s^k,A_t^k\in D^k$ have the same first $k-1$ coordinates, we take $\varphi:A_s^k\to A_t^k$ which replaces $\varepsilon_i(j)=\varepsilon_i^{(s)}(j)$ by $\varepsilon_i^{(t)}(j)$ and interchanges $\pi_i(j)=\pi_i^{(s)}(j)$ with $\pi_i^{(t)}(j)$, the change affecting at most two coordinates.

Thus, if $h_nm_n\geqq n$ and $\Omega_n=E_2^{h_nm_n}\times\Pi_{h_n}^{m_n}$, then $(\Omega_n)_{n=1}^\infty$ is a $\frac{1}{64}$-normal Levy family.

(vi) C_m^n, the space of all (unordered) m-tuples from $\{1,\dots,n\}$ with the normalized Hamming metric, is also in $\mathcal{S}(2,n)$. Once again, D^k is the decomposition determined by the first k coordinates. If A_i^k has 0 at the k-th coordinate and A_j^k has 1, then $\varphi_{i,j}^k$ puts 1 at the k-th coordinate and replaces the first nonzero coordinate after the k-th by a zero. Thus, $(C_m^n)_{n=1}^\infty$ is a $\frac{1}{64}$-normal Levy family.

(vii) The same martingale argument as in [13] shows that if (Ω,d,μ) is any n.m.p.s., then the product spaces Ω^n with the l_1-sum metric

$$d_n^1(s,t)=\frac{1}{n}\sum_{i=1}^n d(s_i,t_i)$$

and the product measure μ^n, form a $\frac{1}{16}$-normal Levy family. Indeed, letting F_k $(k=0,\dots,n)$ be the field generated by the first k coordinates, we get for every $f:\Omega^n\to R$ satisfying $|f(s)-f(t)|\leqq d_n^1(s,t)$,

$$|\mathbf{E}(f\mid F_k)-\mathbf{E}(f\mid F_{k-1})|\leqq\frac{1}{n}\quad\text{hence}\quad\mu^n\{|f-\mathbf{E}f|\geqq\delta\}\leqq 2\exp(-\delta^2 n/4)\quad\text{etc.}$$

2.3. The Levy-function estimates enable us to show concentration in measure of functions: If $\alpha_\Omega(\delta) \leqq \alpha_0 e^{-\tau\delta^2}$ and $f:\Omega \to \mathbf{R}$ has modulus of continuity $\omega_f(\delta)$, consider a *Levy mean* (median) M_f of f, i.e., such that $\mu\{t; f(t) \leqq M_f\} \geqq \frac{1}{2}$ and $\mu\{t; f(t) \geqq M_f\} \geqq \frac{1}{2}$. Then

$$\mu\{t; |f(t) - M_f| \leqq \omega_f(\delta)\} \geqq 1 - 2\alpha_\Omega(\delta).$$

Thus, given N such functions $f_1, \ldots, f_N$, we have for every $\delta > 0$

$$\mu(\{t \in \Omega; |f_i(t) - M_{f_i}| \leqq \omega_{f_i}(\delta), i = 1, \ldots, N\}) \geqq 1 - 2N\alpha_\Omega(\delta).$$

2.4. The Levy mean is close to the average $\mathbf{E}f = \int_\Omega f(t)d\mu$: For every $\delta > 0$ we have

$$|M_f - \mathbf{E}f| \leqq \omega_f(\delta) + 2\omega_f(1)\alpha_\Omega(\delta)$$

([2], p. 9).

2.5. Let $F = \{f_i\}_{i=1}^N$ be a family of N functions on (Ω, d, μ). In the further applications it is crucially important to find at least one $t_0 \in \Omega$ (the same for all the family $\{f_i\}_{i=1}^N$) such that every $f_i(t_0)$ is close enough to its average $\mathbf{E}_{f_i}$. Of course, some conditions should be imposed on the continuity of the f_i, e.g. Hölder-continuity. Combining 2.3 and 2.4 leads to such a result.

Proposition. *If (Ω, d, Ω) is an n.m.p.s. with $\alpha_\Omega(\delta) \leqq \alpha_0 e^{-\tau\delta^2}$ and $f_i : \Omega \to \mathbf{R}$, $i = 1, \ldots, N$ satisfy $|f_i(s) - f_i(t)| \leqq Cd(s,t)^\gamma$ (for $i = 1, \ldots, N$ and all s, $t \in \Omega$), then a sufficient condition for the existence of $t \in \Omega$ with $|f_i(t) - \mathbf{E}f_i| < \varepsilon$ for $i = 1, \ldots, N$ is*

$$C < \frac{\varepsilon}{3} \min\left(\tfrac{1}{3}N, \left(\frac{\tau}{\log 2\alpha_0 N}\right)^{\gamma/2}\right).$$

Proof. If

$$\delta > \left(\frac{\log 2\alpha_0 N}{\tau}\right)^{1/2}$$

yet $C\delta^\gamma < \varepsilon/3$, then by 2.3

$$\mu\left(\left\{t \in \Omega; |f_i(t) - M_{f_i}| < \frac{\varepsilon}{3}, i = 1, \ldots, N\right\}\right) > 1 - 2N\alpha_0 e^{-\tau\delta^2} > 0,$$

while by 2.4,

$$|M_{f_i} - \mathbf{E}_{f_i}| < \frac{\varepsilon}{3} + 2C\alpha_0 e^{-\tau\delta^2} < \tfrac{2}{3}\varepsilon.$$

3. The finite-dimensional Krivine theorem

Applying the results of Section 2 to normed linear spaces, we shall use the following two simple lemmas:

3.1. LEMMA. *If* $\|\cdot\|$, $|||\cdot|||$ *are two norms on a linear space so that* $|\,\|x\| - |||x|||\,| \leqq \varepsilon/10$ *for all* x *in an* $\varepsilon/10$*-net of the* $|||\cdot|||$*-unit sphere, then* $\|\quad\|$ *and* $|||\cdot|||$ *are* $(1+\varepsilon)$*-isomorphic.*

(For the straightforward proof cf., e.g., the proof of Theorem 2 in [2].)

3.2. LEMMA. *For every norm on* $\mathbf{R}^m$ *and every* $\varepsilon > 0$, *there is an* ε*-net* $x_1, \ldots, x_N$ *for the unit sphere with* $N < (1+2/\varepsilon)^m$ ([3], Lemma 2.5).

3.3. The following finite-dimensional version of Krivine's construction of almost l_p^N subspaces from symmetric sequences is given in [2] (Theorem 3.1).

THEOREM. *If* $(y_1, \ldots, y_n)$ *is a symmetric finite sequence in a normed space* X *satisfying*

$$(*) \qquad C_1^{-1}\left(\sum_{i=1}^{n} |\alpha_i|^p\right)^{1/p} \leqq \left\|\sum_{i=1}^{n} \alpha_i y_i\right\| \leqq C_2\left(\sum_{i=1}^{n} |\alpha_i|^p\right)^{1/p}$$

for every $\alpha_1, \ldots, \alpha_n$, *then for every* $\varepsilon > 0$ *there is a block sequence* $(u_1, \ldots, u_k)$ *with respect to* $(y_1, \ldots, y_n)$ *with*

$$(**) \quad (1-\varepsilon)\left(\sum_{j=1}^{k} |\alpha_j|^p\right)^{1/p} \leqq \left\|\sum_{j=1}^{k} \alpha_j u_j\right\| \leqq (1+\varepsilon)\left(\sum_{j=1}^{k} |\alpha_j|^p\right)^{1/p} \quad \textit{for every } \alpha_1, \ldots, \alpha_k,$$

where

$$k \geqq \Gamma^{3p} n^{\Gamma}, \qquad \Gamma = \left(\frac{\varepsilon}{36 C_1 C_2}\right)^p.$$

(In fact, the u_i are equally distributed and are constructed as a "geometric series" — a normalization of a disjoint sum of h bumps of length $((a+1)/a)^j$ and height $((a+1)/a)^{(h-j)/p}$, where a, h are specially chosen integers.)

3.4. By 3.3, our task in the quantitative Krivine theorem reduces to getting "good" symmetric sequences in isomorphs of l_p.

THEOREM. *If the sequence* $(x_1, \ldots, x_n)$ *in a normed space satisfies* $(*)$ *above then, for every* $\varepsilon > 0$, *it has a block sequence* $(y_1, \ldots, y_k)$ *satisfying* $(**)$, *where*

$$k \sim \kappa(\varepsilon, C_1 C_2, p) n^{\Gamma/3}, \qquad \Gamma = \left(\frac{\varepsilon}{36 C_1 C_2}\right)^p$$

and the function κ *is easily computed from the estimate on* m *below and Theorem* 3.3.

PROOF. Partition $(x_1,\dots,x_n)$ into m subsequences of h elements each, $(x_{1,1},x_{1,2},\dots,x_{1,h}),(x_{2,1},\dots,x_{2,h}),\dots,(x_{m,1},\dots,x_{m,h})$ and define, for $\boldsymbol{a}\in\mathbf{R}^m$, $\boldsymbol{t}\in(E_2^m)^h$, $\boldsymbol{\pi}\in(\Pi_m)^h$:

$$\varphi_a(\boldsymbol{t},\boldsymbol{\pi})=\left\|\sum_{i=1}^{h}\sum_{j=1}^{m}t_i(j)a_{\pi_i(j)}x_{i,j}\right\|.$$

Let $|||\,\boldsymbol{a}\,|||=\mathbf{E}(\varphi_a)$ (over $E_2^{mh}\times\Pi_m^h$). Then

$$|||\,a\,|||\geqq\frac{1}{C_1}h^{1/p}\|\boldsymbol{a}\|_p,$$

so that if $|||\,\boldsymbol{a}\,|||=1$ then $\|\boldsymbol{a}\|_\infty\leqq\|\boldsymbol{a}\|_p\leqq C_1h^{-1/p}$ and therefore

$$\begin{aligned}\omega_{\varphi_a}(\delta)&=\sup\left\{\left\|\sum_{i=1}^{h}\sum_{j=1}^{m}t_i(j)a_{\pi_i(j)}x_{i,j}-t_i'(j)a_{\pi_i'(j)}x_{i,j}\right\|;\right.\\&\qquad\left.(t_i(j),\pi_i(j))\neq(t_i'(j),\pi_i'(j))\text{ at most }hm\delta\text{ times}\right\}\\&\leqq2\sup\left\{\left\|\sum_{\nu=1}^{hm\delta}\alpha_{i_\nu}x_{i_\nu}\right\|;\ \max_\nu|\alpha_{i_\nu}|\leqq C_1h^{-1/p}\right\}\\&\leqq2C_2\sup\left\{\left(\sum_{\nu=1}^{hm\delta}|\alpha_{i_\nu}|^p\right)^{1/p};\ \max_\nu|\alpha_{i_\nu}|\leqq C_1h^{-1/p}\right\}\\&=2C_1C_2(m\delta)^{1/p}.\end{aligned}$$

Let $\boldsymbol{a}_\nu$, $\nu=1,\dots,N$, $N=\frac{1}{2}(25/\varepsilon)^m$ be an $\varepsilon/10$-net for the $|||\cdot|||$-unit sphere. To find $(\boldsymbol{t},\boldsymbol{\pi})\in E_2^{mh}\times\Pi_m^h$ with $|\varphi_{a_\nu}(\boldsymbol{t},\boldsymbol{\pi})-1|\leqq\varepsilon/10$ for all ν it suffices, by 2.5 and 2.2(v), that

$$2C_1C_2m^{1/p}<\frac{\varepsilon}{30}\min\left(\frac{1}{6}\left(\frac{25}{\varepsilon}\right)^m,\left(\frac{hm}{64m\log\frac{25}{\varepsilon}}\right)^{1/2p}\right)$$

(since in our case $\alpha_0=1$ and $\tau=hm/64$), hence that

$$m^{1/p}<\frac{\varepsilon}{60C_1C_2}\min\left(\frac{1}{6}\left(\frac{25}{\varepsilon}\right)^m,\left(\frac{\varepsilon h}{1600}\right)^{1/2p}\right),$$

and this is satisfied if, e.g.,

$$m=\tfrac{1}{12}\varepsilon^{(2p+1)/3}(60C_1C_2)^{-2p/3}n^{1/3}$$

(and $25^m>360C_1C_2m$). Since $|||\cdot|||$ is symmetric, $z_\nu=\sum_{i=1}^{h}t_i(\pi_i^{-1}\nu)x_{i,\pi_i^{-1}(\nu)}$ $\nu=1,\dots,m$ is $(1+\varepsilon)$-symmetric by Lemma 3.1.

Applying Theorem 3.3 (after $(1+\varepsilon)$-change of the norm) we get $y_1,\dots,y_k$ satisfying (**).

3.5. This result is "almost" exact in the following sense: we cannot get here a power $k = n^\alpha$ with α not dependent on ε or on $C = C_1C_2$.

EXAMPLE. Let $p_n \leqq q_n \leqq 2$ satisfy

$$\frac{1}{p_n} - \frac{1}{q_n} = \frac{\log C}{\log n}.$$

Then by [6] we know that $d(l_{p_n}^n, l_{q_n}^n) = C$. If E is *any* n^α-dimensional subspace of $l_{q_n}^n$ then, by a result of Lewis [8],

$$d(E, l_2^{n^\alpha}) \leqq n^{\alpha(q_n^{-1} - 1/2)},$$

while, by [6],

$$d(l_{p_n}^{n^\alpha}, l_2^{n^\alpha}) = n^{\alpha(p_n^{-1} - 1/2)}.$$

Hence

$$d(E, l_{p_n}^{n^\alpha}) \geqq n^{\alpha(p_n^{-1} - q_n^{-1})} = C^\alpha,$$

and if we want it to be $\leqq 1 + \varepsilon < e^\varepsilon$, we must have $\alpha < \varepsilon / \log C$ (similar reasoning was used in [1]).

4. Symmetric sets in the range

4.1. Another application of measure concentration is to find large "almost symmetric" sets in the range of Lipschitz–Hölder functions on an n.m.p.s. (Ω, d, μ). By 2.2 (vii), the powers $(\Omega^n)_{n=1}^\infty$ form a $\frac{1}{16}$-normal Levy family in the l_1-sum metric d_1^n. It may happen that $(\Omega^n)_{n=1}^\infty$ is a normal Levy family even in the weaker l_r-sum metric,

$$d_r^n(s, t) = m^{-1/r} \left(\sum_{i=1}^m d_i(s_i, t_i)^r \right)^{1/r}$$

for some $r \geqq 1$ (as in the case 2.2(ii) of $\Omega = S_{m-1}$, where $r = 2$). In fact the most natural applications of the following proposition are to the cases $\Omega = E_2^m$, $r = 1$ and $\Omega = S_{m-1}$, $r = 2$; see 4.3(i) and (ii).

4.2. PROPOSITION. *Let (Ω, d, μ) be a compact n.m.p.s. with a measure-preserving isometric fixed-point free involution $t \to -t$ and such that, for some $r \geqq 1$ and τ, $(\Omega^m)_{m=1}^\infty$ is a τ-normal Levy family with the l_r-sum metric and the product measure μ^m. Let $f : \Omega \to X$ be a non-0 odd function (i.e., with $f(-t) = -f(t)$) satisfying $\|f(s) - f(t)\| \leqq Cd(s, t)^\gamma$ for some $0 < \gamma \leqq 1$, and let C_q ($2 \leqq q \leqq$*

∞) be the q-Rademacher cotype constant of the normed linear space X ($C_q = \sup(\Sigma_{i=1}^n \|x_i\|^q)^{1/q}$; $x_i \in X$, $(\mathbf{E}(\|\Sigma_{i=1} \varepsilon_i x_i\|^q)^{1/q} \leqq 1)$ if $2 \leqq q < \infty$, $C_\infty = 1$). Then, for every $\varepsilon > 0$, a $(1+\varepsilon)$-symmetric m-tuple exists in the range of f, where

$$m = \left(\frac{\varepsilon \mathbf{E}(\|f\|)}{30CC_q}\left(\frac{\varepsilon\tau}{20}\right)^{\gamma/2}\right)^{\min(q/(q-1), r/\gamma)}$$

PROOF (a modification of the proof of Theorem 2.2 in [2]). Since Ω is compact, we can specify a "positive half" Ω^+ of Ω so that $\mu(\Omega^+) = \frac{1}{2}$ and $\Omega^+ \cap (-\Omega^+) = \emptyset$. For every $\boldsymbol{a} \in \mathbf{R}^m$ define the function $\varphi_a(t) = \|\Sigma_{j=1}^m a_j f(t_j)\|$ on Ω^m, and then define on $\mathbf{R}^m$:

$$|||\,\boldsymbol{a}\,||| = \mathbf{E}(\varphi_a) = \int_\Omega \cdots \int_\Omega \left\| \sum_{j=1}^m a_j f(t_j) \right\| d\mu(t_1) \cdots d\mu(t_m).$$

$|||\cdot|||$ is clearly a semi-norm on $\mathbf{R}^m$. It is symmetric, since if $\boldsymbol{\varepsilon} \in E_2^m$ and $\pi \in \Pi_m$ then

$$\begin{aligned} |||(\boldsymbol{\varepsilon}, \pi)(\boldsymbol{a})||| &= \int_\Omega \cdots \int_\Omega \left\| \sum_{j=1}^m \varepsilon_j a_{\pi(j)} f(t_j) \right\| d\mu(t_1) \cdots d\mu(t_m) \\ &= \int_\Omega \cdots \int_\Omega \left\| \sum_{j=1}^m a_j f(\varepsilon_{\pi^{-1}j} t_{\pi^{-1}j}) \right\| d\mu(t_1) \cdots d\mu(t_m) \\ &= \int_\Omega \cdots \int_\Omega \left\| \sum_{j=1}^m a_j f(s_j) \right\| d\mu(s_1) \cdots d\mu(s_m) \\ &= |||\,\boldsymbol{a}\,|||. \end{aligned}$$

Since $|||\cdot|||$ is unconditional,

$$|||\,\boldsymbol{a}\,||| \geqq \max_j |||\,a_j e_j\,||| = \max_j |a_j| \mathbf{E}(\|f\|),$$

and $|||\cdot|||$ is a norm. We also have, for every $1 \leqq q < \infty$,

$$\begin{aligned} |||\,\boldsymbol{a}\,||| &= \int_{\Omega^+} \cdots \int_{\Omega^+} \sum_{\varepsilon \in E_2^m} \left\| \sum_{j=1}^m \varepsilon_i a_j f(t_j) \right\| d\mu(t_1) \cdots d\mu(t_m) \\ &\geqq \frac{2^m}{C_q} \int_{\Omega^+} \cdots \int_{\Omega^+} \left(\sum_{j=1}^m |a_j|^q \|f(t_j)\|^q \right)^{1/q} d\mu(t_1) \cdots d\mu(t_m) \\ &= \frac{1}{C_q} \int_\Omega \cdots \int_\Omega \left(\sum_{j=1}^m |a_j|^q \|f(t_j)\|^q \right)^{1/q} d\mu(t_1) \cdots d\mu(t_m) \\ &\geqq \frac{1}{C_q} \|\boldsymbol{a}\|_q \mathbf{E}(\|f\|). \end{aligned}$$

Therefore, if $|||\, \boldsymbol{a}\, ||| = 1$ then $\|\boldsymbol{a}\|_q \leqq C_q/\mathbf{E}(\|f\|)$ and, for every $\boldsymbol{s}, \boldsymbol{t} \in \Omega^m$, we have

$$|\varphi_a(s) - \varphi_a(t)| \leqq \sum_{j=1}^m |a_j|\, \|f(s_j) - f(t_j)\|$$
$$\leqq \|\boldsymbol{a}\|_q \left(\sum_{j=1}^m \|f(s_j) - f(t_j)\|^{q/(q-1)} \right)^{(q-1)/q}$$
$$\leqq \frac{CC_q}{\mathbf{E}(\|f\|)} \left(\sum_{j=1}^m d(s_j, t_j)^{\gamma q/(q-1)} \right)^{(q-1)/q}.$$

Since $\sum_{j=1}^m d_j^\alpha$ $(0 < \alpha \leqq 2)$ constrained by $d_j \geqq 0$, $\sum_{j=1}^m d_j^r = m\delta^r$ $(r \geqq 1)$ attains its maximum when $d_j = \delta$ $(j = 1, \dots, m)$ if $\alpha \leqq r$, and when $d_1 = m^{1/r}\delta$, $d_j = 0$ $(j = 2, \dots, m)$ if $\alpha \geqq r$, we get the estimates:

$$\omega_{\varphi_a}(\delta) \leqq \frac{CC_q}{\mathbf{E}(\|f\|)}\, m^{(q-1)/q}\delta^\gamma \qquad \text{if } \frac{\gamma q}{q-1} \leqq r,$$
$$\omega_{\varphi_a}(\delta) \leqq \frac{CC_q}{\mathbf{E}(\|f\|)}\, m^{\gamma/r}\delta^\gamma \qquad \text{if } \frac{\gamma q}{q-1} \geqq r$$

(for $q = \infty$ we simply have

$$\omega_{\varphi_a}(\delta) \leqq \frac{Cm}{\mathbf{E}(\|f\|)}\, \delta^\gamma \Big).$$

Take an $\varepsilon/10$-net for the $|||\cdot|||$-unit sphere, $(\boldsymbol{a}_\nu)_{\nu=1}^N$, with $2N \sim (30/\varepsilon)^m$. Since $E(\varphi_{a_\nu}) = |||\, \boldsymbol{a}\, ||| = 1$, we can apply 2.5 to get $\boldsymbol{t} \in \Omega^m$ so that $|\varphi_{a_\nu}(\boldsymbol{t}) - 1| \leqq \varepsilon/10$ for all $\nu = 1, \dots, N$, provided that

$$\frac{CC_q}{\mathbf{E}(\|f\|)}\, m^{\max((q-1)/q, \gamma/r)} < \frac{\varepsilon}{30}\left(\frac{\varepsilon\tau}{30}\right)^{\gamma/2} \qquad (\text{and } m > \log 2\alpha_0\tau).$$

4.3. (i) In the special case $f: S_{k-1} \to S(X)$ (the unit sphere of X) in which $r = 2$, $\tau = \pi^2 k/2$, Proposition 4.1 yields a $(1+\varepsilon)$-symmetric sequence of length $\theta\varepsilon^3(CC_q)^{(1-q)/q}k^{\gamma q/2(q-1)}$ ([2], Theorem 2.2).

(ii) A similar estimate is obtained for $f: E_2^k \to S(X)$. This time $r = 1$, $\tau = 2k$, hence we get a $(1+\varepsilon)$-symmetric sequence of length $\theta\varepsilon^3(CC_q)^{-2}k^{\min(1/2, \gamma q/2(q-1))}$.

4.4. Corollary. *If $(x_1, \dots, x_k)$ is a sequence in the unit ball of the normed linear space X such that $E(\|\sum_{i=1}^k \varepsilon_i x_i\|) \geqq \theta k^{1/p}$ for some $1 \leqq p < 2$, then there are m choices of signs $\varepsilon^j \in E_2^m$ so that the sequence $y_j = \sum_{i=1}^k \varepsilon_i^j x_i$, $j = 1, \dots, m$ is $(1+\varepsilon)$-symmetric where $m = ck^{1/p-1/2}$, $c = \varepsilon^{3/2}\theta/200$.*

Proof. Consider $f: E_2^k \to X$ defined by $f(t) = \sum_{i=1}^k t_i x_i$. By the triangle inequality, $\|f(s)-f(t)\| \leqq 2kd(s,t)$. By Proposition 4.2 with $q=\infty$, $C_q=1$, we can take

$$m=\frac{\varepsilon\theta k^{1/p}}{30\cdot 2k}\left(\frac{\varepsilon k}{10}\right)^{1/2}=\varepsilon^{3/2}\frac{\theta k^{1/p-1/2}}{60\sqrt{10}}.$$

This result which we have for a "linear type" sequence by a "nonlinear" general approach is not the best possible. In this case the methods of Johnson and Schechtman and Pisier yield better results (cf. [11]).

4.5. Unfortunately, one cannot get a "good" $n=n(k,C,\gamma)$ so that an f as in 4.3(i) or (ii) will exist for all n-dimensional spaces X. This is shown by the following argument (cf. [5]): Let $X=l_\infty^n$, and f as in 4.3. Then each coordinate $f_i(t)=(f(t))_i$ is odd, hence $\mathbf{E}f_i=M_{f_i}=0$. It satisfies also the same Lipschitz–Hölder estimate, hence by 2.5,

$$\mu\{t;|f_i(t)|<1,\ i=1,\dots,n\}>0 \qquad \text{if } C<\tfrac{1}{3}\min\left(\frac{n}{3},\left(\frac{\tau}{\log 2\alpha_0 n}\right)^{\gamma/2}\right).$$

This cannot happen since $\max_{1\leqq i\leqq n}|f_i(t)|=1$ for all $t\in\Omega$. Thus, for $n>10C$ we must have $\tau<(3C)^{2/\gamma}\log 2\alpha_0 n$. But $\tau=\theta k$ for some θ, hence we must have

$$n(k,C)>\frac{1}{2\alpha_0}\exp(\theta_1 kC^{-2/\gamma}).$$

5. Symmetric block sequences

5.1. The measure concentration argument in E_2^n was applied in [2], Theorem 2.3 to get from a "type attaining" sequence in a normed space X an "almost unconditional" block sequence and then, in Theorem 2.4, the measure concentration argument in Π_n was used to get an "almost symmetric" block sequence. This could have been done in one step using the measure concentration in the product space $E_2^{nk}\times\Pi_n^k$, improving the estimate from $cn^{(2-p)/3p(2+p)}$ to $c_1n^{(1-p)/p(2+p)}$ (for details, see: D. Amir, *Some applications of concentration phenomena*, Longhorn Notes, The University of Texas Functional Analysis Seminar, 1982–1983, pp. 161–178).

5.2. There is a calculational mistake in Theorem 2.5 of [2]. We shall give a "more correct" version of it here.

THEOREM. *Let $p\in(1,2)$, $\theta,\varepsilon\in(0,1)$. If $(x_1,\dots,x_n)$ is a sequence of norm-1 elements in a normed space x such that $\mathbf{E}(\|\sum_{i=1}^n\varepsilon_i x_i\|)\geqq\theta n^{1/p}$, then there is a*

$(1+\varepsilon)$-symmetric sequence $\{y_1,\dots,y_m\}$ of blocks with disjoint support and ± 1 coefficients and of length

$$m=\frac{1}{500}\varepsilon^{3/2}n^{(2-p)^2/3p^3}.$$

PROOF. Let

$$\beta=\frac{p}{2}+\frac{(2-p)^2}{2p},\qquad \alpha=\frac{2p\beta}{2\beta+p}.$$

Then $0<\alpha<\beta<1$. As in the proof of Theorem 2.5 in [2] we get a subset, which we may assume to be $x_1,\dots,x_{km}$, $km=n^{1/p}$, such that

$$\mathbf{E}\left(\left\|\sum_{x_\nu\in A}\varepsilon_\nu x_\nu\right\|\right)\geqq|A|^{\beta/p}$$

for each of its subsets A of length $|A|\geqq(km)^{\alpha/\beta}$ (in particular, by our choice of m, k, α and β, when $|A|\geqq k$, provided $n>n_0(\varepsilon,p)$).

For $\boldsymbol{a}\in\mathbf{R}^m$, $(\boldsymbol{t},\boldsymbol{\pi})\in E_2^{km}\times\Pi_{km}$ define

$$\varphi_{\boldsymbol{a}}(\boldsymbol{t},\boldsymbol{\pi})=\left\|\sum_{j=1}^m a_j\sum_{i=(j-1)k+1}^{jk}t_ix_{\pi(i)}\right\|,$$

and then $|||\,\boldsymbol{a}\,|||=\mathbf{E}(\varphi_{\boldsymbol{a}})$. We have $|||\,\boldsymbol{a}\,|||\geqq k^{\beta/p}\|\boldsymbol{a}\|_\infty$ hence, if $|||\,\boldsymbol{a}_\nu\,|||=1$, then $\omega_{\varphi_{\boldsymbol{a}_\nu}}(\delta)\leqq 2mk^{1-\beta/p}\delta$. Taking an $\varepsilon/10$-net for the $|||\cdot|||$-sphere, $\boldsymbol{a}_1,\dots,\boldsymbol{a}_N$, $N\sim(20/\varepsilon)^m$, $(\boldsymbol{t},\boldsymbol{\pi})\in E_2^{km}\times\Pi_{km}$ satisfying

$$|\varphi_{\boldsymbol{a}_\nu}(\boldsymbol{t},\boldsymbol{\pi})-1|\leqq\varepsilon/10\qquad\text{for }\nu=1,\dots,N$$

exists provided

$$2mk^{1-\beta/p}<\frac{\varepsilon}{30}\left(\frac{k\varepsilon}{64}\right)^{1/2},$$

hence if

$$m<\frac{1}{480}\varepsilon^{3/2}k^{(2-p)^2/2p^2},$$

which holds in our case (we have to check that $n^2<k^{3p}$, i.e., that $m^{3p}<n$). By Lemma 3.1,

$$y_j=\sum_{i=(j-1)k+1}^{jk}t_ix_{\pi(i)},\qquad j=1,\dots,m,$$

is $(1+\varepsilon)$-symmetric.

REMARK. We also get an estimate on the characteristic function $\lambda(\nu) = \|\Sigma_{i=1}^{\nu} y_i\|$ of the (almost) symmetric sequence $(y_i)_{i=1}^m : \lambda(\nu) \geqq \eta(k\nu)^{\beta/p}$.

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